Comment on "Energy Dissipation in an Oscillating Sphere Filled with a Viscous Fluid"

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In Ref. 1 the authors claim to have generated an exact solution of the Navier-Stokes equation, corresponding to the motion of a viscous incompressible fluid inside a hollow sphere set up by oscillations of the sphere about a diameter. Unfortunately, their result is not in fact an exact solution of the Navier-Stokes equation. For, if one follows the authors of Ref. 1 and, using their coordinate system, sets radial and meridional velocity components identically equal to zero, and takes the azimuthal velocity component $v_\phi = \psi(r, \theta, t)$, the complete Navier-Stokes equation,[2] which expresses conservation of momentum in all three coordinate directions, implies the three equations

$$\psi^2/r = (1/\rho)(\partial\hat{p}/\partial r) \tag{1}$$

$$\psi^2 \cot\theta/r = (1/\rho)(1/r)(\partial\hat{p}/\partial\theta) \tag{2}$$

$$\frac{\partial\psi}{\partial t} = \nu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) - \frac{\psi}{r^2\sin^2\theta}\right] \tag{3}$$

Equation (3) is seen to be Eq. (1) of Ref. 1, after the correction of what seems to be a printing error in the latter. The authors of Ref. 1 obtained a solution of this equation, but they neglected Eqs. (1) and (2) above. As it happens, there is no pressure function $\hat{p}(r, \theta, t)$ such that Eqs. (1) and (2) can be satisfied for the function ψ determined in Ref. 1. For, eliminating $\hat{p}$ between (1) and (2), one finds that these equations are inconsistent unless

$$(\partial/\partial\theta)(\psi^2/r) = (\partial/\partial r)(\psi^2 \cot\theta)$$

from which it follows that

$$(\partial\psi/\partial\theta) = r\cot\theta(\partial\psi/\partial r) \tag{4}$$

One can verify by substitution that the result of Ref. 1 does not satisfy Eq. (4). More simply but less directly, one can observe that the solution ψ found in Ref. 1 has the form

$$\psi = \sin\theta f_1(r, t)$$

and that a function of this form can satisfy Eq. (4) only if $f_1(r, t) = C(t)r$. This is not the $f_1(r, t)$ determined in Ref. 1. Thus, the result of Ref. 1 is not an exact solution of the Navier-Stokes equation; indeed, the velocity field determined there does not conserve momentum in the radial and meridional directions.

Nevertheless, the work of Ref. 1 can be useful, if it is properly interpreted. This is indicated by the quite acceptable agreement between experiment and theory shown in Fig. 2 of Ref. 1. In fact, Ref. 1 can be interpreted as a treatment of the first-order terms in a formal expansion of the velocity field in powers of Φ, the amplitude of the sphere's angular displacement. In Ref. 1, the motion of the sphere was characterized by stating that its angular velocity Ω was sinusoidal in time, i.e.,

$$\Omega = \Omega_0 \cos pt$$

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This is of course equivalent to specifying the sphere's angular displacement $\hat{\phi}$ to be sinusoidal,

$$\hat{\phi} = \Phi \sin pt$$

In terms of the sphere's angular displacement, the boundary condition at the surface of the sphere is

$$v_\phi(a, \theta, t) = a\Omega\sin\theta = ap\Phi\sin\theta\cos pt$$

with all other velocity components zero. If the problem is nondimensionalized, using pa as reference velocity and a as a reference length, this condition becomes

$$v_\phi'(1, \theta, t') = \Phi\sin\theta\cos t' \tag{5}$$

where here and in the following a prime denotes a nondimensional variable. If now one enters the nondimensionalized Navier-Stokes equation, the conservation of mass condition of a solenoidal velocity field, and boundary condition (5) with the formal expansion

$$v_r' = \Phi^2 u_2 + \Phi^3 u_3 + \ldots \tag{6a}$$

$$v_\theta' = \Phi^2 v_2 + \Phi^3 v_3 + \ldots \tag{6b}$$

$$v_\phi' = \Phi w_1 + \Phi^2 w_2 + \ldots \tag{6c}$$

$$(p/\rho) = \Phi^2 \pi_2 + \Phi^3 \pi_3 + \ldots \tag{6d}$$

where each dependent variable is taken to be a function of r', θ, and t', one finds that this expansion is not obviously inconsistent, and that the lowest order terms, i.e., those of order Φ, imply

$$\frac{\partial w_1}{\partial t'} = \frac{1}{\gamma}\left[\frac{1}{r'^2}\frac{\partial}{\partial r'}\left(r'^2\frac{\partial w_1}{\partial r'}\right) + \frac{1}{r'^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial w_1}{\partial\theta}\right) - \frac{w_1}{r'^2\sin^2\theta}\right],\quad w_1(1, \theta, t') = \sin\theta\cos t'$$

where, as in Ref. 1, $\gamma = pa^2/\nu$. In essence, this is the problem solved in Ref. 1.

Thus, while of course the convergence of expansion (6) requires investigation, the work of Ref. 1 may well be accurate for the limiting case in which the amplitude Φ of the sphere's oscillation tends to zero while γ is fixed. This seems to have been the application its authors actually had in mind. But, since the results of Ref. 1 are not really based on an exact solution of the Navier-Stokes equation, it may be dangerous to apply them in other situations.

References

[1] Aprahamian, R., Johnson, R. L., and Koval, L. R., "Energy Dissipation in an Oscillating Sphere Filled with a Viscous Fluid," *AIAA Journal*, Vol. 7, No. 9, Sept. 1969, pp. 1793–1796.

[2] Landau, L. D. and Lifshitz, E. M., *Fluid Mechanics*, Addison Wesley, Reading, Mass., 1959, p. 52.

Reply by Authors to D. A. Lee

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The authors wish to thank D. A. Lee for his interesting comments regarding Ref. 1. It is true as D. A. Lee states that the solution given in Ref. 1 does not conserve momentum

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in either the radial or meridional directions and consequently cannot be an exact solution to the Navier-Stokes equations for the motion of a body of fluid within a sphere rotating about a diameter. Our statement that the solution is "exact" was made unwittingly. It is clear that what was attempted in Ref. 1 was a solution to the motion of the fluid neglecting the effects of centrifugal forces. Since these forces develop radial and meridional pressure gradients the result can in no way satisfy Eqs. (1) and (2) of Ref. 2.

However, it was expected on physical grounds, that the result obtained would be a good approximation when the ratio of angular velocity amplitude to frequency is small, a surmise that was fully justified by the experimental results. Essentially, D. A. Lee arrives at the same conclusion on the basis of more careful mathematical analysis, an effort for which he is to be commended.

One further remark seems in order. Referring to D. A. Lee's comment, Eq. (6), it appears that the satisfaction of these equations to second order in the parameter ϕ, implies the existence of radial and meridional velocity components. If so, then no solution of higher order in the parameter ϕ than that presented by the authors exists, for which only the azimuthal velocity component is nonzero. In other words, the only solution to the problem (as posed by the authors) is that presented in Ref. 1. This can be seen directly from the momentum Eqs. (1), (2), and (3) of D. A. Lee's paper. If the function ψ obtained by solving Eq. (3) is used to obtain a first approximation to the pressure gradient using Eqs. (1) and (2), use of the second-order pressure function so obtained to calculate an improved function ψ is not possible since the equations are not coupled through the pressure. A new value can be obtained only if the radial and meridional velocity components are allowed to take nonzero values. It thus appears that the first-order solution is the only solution corresponding to flow with one velocity component.

References

[1] Aprahamian, R., Johnson, R. L., and Koval, L. R., "Energy Dissipation in an Oscillating Sphere Filled with a Viscous Fluid," *AIAA Journal*, Vol. 7, No. 9, Sept. 1969, pp. 1793–1796.

[2] Lee, D. A., "Comment on 'Energy Dissipation in an Oscillating Sphere Filled with a Viscous Fluid,'' *AIAA Journal*, Vol. 8, No. 10, Oct. 1970, pp. 1919.